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Harmonic Mean and Contra-Harmonic Mean Derivative-Based Closed Newton-Cotes Quadrature

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ABSTRACT

In this paper, the Harmonic mean and Contra-harmonic mean of two endpoints [a,b] are applied at the derivative of the error term in the existing CNC formulas. This modified error term is included as an additional function to the existing formula. The new error terms are derived using the method of precision for the proposed formulas. Comparisons are made between the existing CNC formulas and the proposed CNC formulas by using numerical examples. The proposed Harmonic mean derivate-based (HMD) and the Contra-harmonic mean derivative-based (CHMD). CNC formulas give better results for definite integrals. Another two of its statistical means are discussed and derived for the existing CNC formulas in the subsequent paper.

Keywords- Harmonic mean derivate-based (HMD), Contra-harmonic mean derivative-based (CHMD)

I. INTRODUCTION

The research on derivative-based CNC formulas is continuously increasing. In the previous work, the researchers applied the midpoint or arithmetic mean of endpoints at the derivative of the error term to improve the accuracy of the existing CNC formulas. As a new approach, various statistical means are applied at the error term derivative of the first four CNC formulas. This new method increases a single order of precision over the existing CNC formulas and is used to approximate the definite integral with better accuracy. The previous chapter applies the geometric mean and Heronian mean of end points at the error term derivative of CNC formulas. The proposed formulas also give better results.

In this paper, the Harmonic mean and Contra-harmonic mean of two endpoints [a,b] are applied at the derivative of the error term in the existing CNC formulas. This modified error term is included as an additional function to the existing formula. The new error terms are derived using the method of precision for the proposed formulas. Comparisons are made between the existing CNC formulas and the proposed CNC formulas by using numerical examples. The proposed Harmonic mean derivate-based (HMD) and the Contra-harmonic mean derivative-based (CHMD) CNC formulas give better results for definite integrals. Another two of its statistical means are discussed and derived for the existing CNC formulas in the subsequent paper.

II. METHODOLOGY

In this paper, Harmonic mean and Contra-harmonic mean of end points [a,b] are applied in the derivative of error term in the first four CNC formulas to obtain the HMD and CHMD based CNC formulas.

The Harmonic mean derivative-based closed Newton-Cotes quadrature (HMD-CNC) formulas are

Trapezoidal formula:

$$\int_a^b f(x)dx \approx \frac{b-a}{2}(f(a)+f(b)) - \frac{(b-a)^2}{12} f^{(2)}\left(\frac{2ab}{a+b}\right), \quad (1.1)$$

Simpson's 1/3rd formula:

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{2ab}{a+b}\right), \quad (1.2)$$

Simpson's 3/8th formula:

$\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right] - \frac{(b-a)^5}{6480}f^{(4)}\left(\frac{2ab}{a+b}\right)$, (1.3)

Boole's formula:

$\int_a^b f(x)dx \approx \frac{b-a}{90}\left[7f(a)+32f\left(\frac{3a+b}{4}\right)+12f\left(\frac{a+b}{2}\right)32f\left(\frac{a+3b}{4}\right)+7f(b)\right] - \frac{(b-a)^7}{1935360}f^{(6)}\left(\frac{2ab}{a+b}\right)$, (1.4)

The Contra-harmonic mean derivative-based closed Newton-Cotes quadrature (CHMD-CNC) formulas are

Trapezoidal formula:

$\int_a^b f(x)dx \approx \frac{b-a}{2}(f(a)+f(b)) - \frac{(b-a)^3}{12}f^{(2)}\left(\frac{a^2+b^2}{a+b}\right)$, (1.5)

Simpson's 1/3rd formula:

$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{(b-a)^5}{2880}f^{(4)}\left(\frac{a^2+b^2}{a+b}\right)$, (1.6)

Simpson's 3/8th formula:

$\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right] - \frac{(b-a)^5}{6480}f^{(4)}\left(\frac{a^2+b^2}{a+b}\right)$, (1.7)

Boole's formula:

$\int_a^b f(x)dx \approx \frac{b-a}{90}\left[7f(a)+32f\left(\frac{3a+b}{4}\right)+12f\left(\frac{a+b}{2}\right)32f\left(\frac{a+3b}{4}\right)+7f(b)\right] - \frac{(b-a)^7}{1935360}f^{(6)}\left(\frac{a^2+b^2}{a+b}\right)$, (1.8)

III. HMD-CNC FORMULAS

A new method of evaluation of Harmonic mean derivative-based CNC formulas is derived to evaluate definite integral. In the HMD-CNC formulas, the Harmonic mean of end points is applied in the error term derivative of Trapezoidal formula, Simpson's 1/3rd formula, Simpson's 3/8th formula and Boole's formula. The Harmonic mean derivative is zero in this method if either a=0 or b=0.

Theorem 1.1

Closed Trapezoidal formula using Harmonic mean derivative is

$$\int_a^b f(x)dx \approx \frac{b-a}{2}(f(a)+f(b)) - \frac{(b-a)^3}{12}f^{(2)}\left(\frac{2ab}{a+b}\right),$$

The precision of this method is 2.

Proof:

For $f(x) = x^2$

The exact value of $\int_a^b x^2dx \approx \frac{1}{3}(b^3-a^3)$; (1.1) $\Longrightarrow -\frac{b-a}{2}(a^2+b^2) - \frac{2(b-a)^3}{12} = \frac{1}{3}f(b^3-a^3)$.

It indicates that the solution is exact. Thus, the precision of the closed Trapezoidal formula with Contra-harmonic mean derivative is 2 whereas the precision of the existing Trapezoidal formula is 1.

Theorem 1.2

Closed Simpson's 1/3rd formula with Harmonic mean derivative is

$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{(b-a)^5}{2880}f^{(4)}(\sqrt{ab})$,

The precision of this method is 4.

Proof:

For $f(x) = x^4$ The exact value of $\int_a^b x^4dx \approx \frac{1}{5}(b^5-a^5)$; (1.2) $\Longrightarrow -\frac{b-a}{6}\left[a^4+4\left(\frac{a+b}{2}\right)^4+b^4\right] - \frac{24(b-a)^5}{2880} = \frac{1}{5}f(b^5-a^5)$.

It indicates that the solution is exact. Thus, the precision of the closed Simpson's 1/3rd formula with Contra-harmonic mean derivative is 4 whereas the precision of the existing Simpson's 1/3rd formula is 3.

Theorem 1.3

Closed Simpson's 3/8th formula with Harmonic mean derivative is

$\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right] - \frac{(b-a)^5}{6480}f^{(4)}\left(\frac{2ab}{a+b}\right)$,

The precision of this method is 4.

Proof:

For $f(x) = x^4$ The exact value of $\int_a^b x^4dx = \frac{1}{5}(b^5-a^5)$; (1.3) $\Longrightarrow -\frac{b-a}{8}\left[a^4+3\left(\frac{2a+b}{3}\right)^4+3\left(\frac{a+2b}{3}\right)^4+b^4\right] - \frac{24(b-a)^5}{6480} = \frac{1}{5}f(b^5-a^5)$.

It indicates that the solution is exact. Thus, the precision of the closed Simpson's 3/8th formula with Contra-harmonic mean derivative is 4 whereas the precision of the existing Simpson's 3/8th formula is 3.

Theorem 1.4

Closed Boole's formula with Harmonic mean derivative is

$\int_a^b f(x)dx \approx \frac{b-a}{90}\left[7f(a)+32f\left(\frac{3a+b}{4}\right)+12f\left(\frac{a+b}{2}\right)+32f\left(\frac{a+3b}{4}\right)+7f(b)\right] - \frac{(b-a)^7}{1935360}f^{(6)}\left(\frac{2ab}{a+b}\right)$,

The precision of this method is 6.

Proof:

For $f(x) = x^6$. The exact value of $\int_a^b x^6dx = \frac{1}{7}(b^7-a^7)$; (1.4)$\Longrightarrow -\left(\frac{b-a}{90}\right)\left[7a^6+32\left(\frac{3a+b}{4}\right)^4+12\left(\frac{a+b}{2}\right)^6+32\left(\frac{a+3b}{4}\right)^6+7b^6\right] + \frac{720(b-a)^7}{1935360} = \frac{1}{7}f(b^7-a^7)$.

It indicates that the solution is exact. Thus, the precision of the closed Boole's formula with Contra-harmonic mean derivative is 6 whereas the precision of the existing Boole's formula is 5.

IV. THE ERROR TERMS OF HMD-CNC FORMULAS

The error of approximation for the method based on the precision of a quadrature formula is obtained by using the difference between the quadrature formula for the monomial $\frac{x^{p+1}}{(p+1)!}$ and the exact result $\frac{1}{(p+1)!}\int_a^b x^{p+1}\,dx$ where p is the precision of the quadrature formula.

Theorem 1.5

Harmonic mean derivative-based closed Trapezoidal formula with the error term is

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\big(f(a)+f(b)\big)-\frac{(b-a)^3}{12}f^{(2)}\left(\frac{2ab}{a+b}\right)-\frac{(b-a)^3}{24(a+b)}f^{(4)}(\xi). \quad (1.9)$$

where $\xi \in$ (a, b). This is the fourth order accurate with the error term

$$E_1[f]=-\frac{(b-a)^5}{24(a+b)}f^{(4)}(\xi).$$

Proof:

The exact value of $\frac{1}{3!}\int_a^b x^3 dx=\frac{1}{24}(b^4-a^4)$;

By using (1.1) $\Longrightarrow \frac{b-a}{12(a+b)}\big((b^3+a^3)(a+b)-(b-a)^2(2ab)\big)$

Hence,

$$E_9[f]=-\frac{1}{24}(b^4-a^4)-\frac{b-a}{12(a+b)}\big((b^3+a^3)(a+b)-2(b-a)^2ab\big)=-\frac{(b-a)^5}{24(a+b)}$$

Hence, the error term is

$$E_9[f]=-\frac{(b-a)^5}{24(a+b)}f^{(4)}(\xi).$$

Theorem 1.6

Harmonic mean derivative-based closed Simpson's 1/3rd formula with the error term is

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{(b-a)^5}{2880}f^{(4)}\left(\frac{2ab}{a+b}\right)-\frac{(b-a)^7}{5760(a+b)}f^{(6)}(\xi). \quad (1.10)$$

where $\xi \in$ (a, b). This is the sixth order accurate with the error term

$$E_{10}[f]=-\frac{(b-a)^7}{5760(a+b)}f^{(6)}(\xi).$$

Proof:

The exact value of $\frac{1}{5!}\int_a^b x^5 dx=\frac{1}{720}(b^6-a^6)$;

By using (1.2) $\Longrightarrow \frac{b-a}{5760(a+b)}\big((8b^5+(a+b)^5+8b^5)(a+b)-4(b-a)^4ab\big)$,

Hence,

$$E_{10}[f]=\frac{1}{720}(b^6-a^6)-\frac{b-a}{5760(a+b)}\big((8b^5+(a+b)^5+8b^5)(a+b)-4(b-a)^4ab\big)=-\frac{(b-a)^7}{5760(a+b)}.$$

Hence, the error term is

$$E_{10}[f]=-\frac{(b-a)^7}{5760(a+b)}f^{(6)}(\xi).$$

Theorem 1.7

Harmonic mean derivative-based closed Simpson's 3/8th formula with the error term is

$$\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{(b-a)^5}{6480}f^{(4)}\left(\frac{2ab}{a+b}\right)-\frac{(b-a)^7}{12960(a+b)}f^{(6)}(\xi). \quad (3.11)$$

Where $\xi \in$ (a, b). This is the sixth order accurate with the error term

$$E_{11}[f]=-\frac{(b-a)^7}{12960(a+b)}f^{(6)}(\xi).$$

Proof:

The exact value of $\frac{1}{5!}\int_a^b x^5 dx=\frac{1}{720}(b^6-a^6)$;

By using (1.3) $\Longrightarrow \frac{b-a}{12960(a+b)}\big((81b^5+(2a+b)^5+(a+2b)^5+81b^5)(a+b)-24(b-a)^4ab\big)$

Hence,

$$E_{11}[f]=\frac{1}{720}(b^6-a^6)-\frac{b-a}{12960(a+b)}\big((81b^5+(2a+b)^5+(a+2b)^5+81b^5)(a+b)-24(b-a)^4ab\big)=-\frac{(b-a)^7}{12960(a+b)}.$$

Hence, the error term is

$$E_{11}[f]=-\frac{(b-a)^7}{12960(a+b)}f^{(6)}(\xi).$$

Theorem 1.8

Harmonic mean derivative-based closed Boole's formula with the error term is

$$\int_a^b f(x)dx \approx \frac{b-a}{90}\left[7f(a)+32f\left(\frac{3a+b}{4}\right)+12f\left(\frac{a+b}{2}\right)+32f\left(\frac{a+3b}{4}\right)+7f(b)\right]-\frac{(b-a)^7}{1935360}f^{(6)}\left(\frac{2ab}{a+b}\right)-\frac{(b-a)^9}{3870720(a+b)}f^{(8)}(\xi). \quad (3.12)$$

where $\xi \in$ (a, b). This is the eighth order accurate with the error term

$$E_{12}[f]=-\frac{(b-a)^9}{3870720(a+b)}f^{(8)}(\xi).$$

Proof:

The exact value of $\frac{1}{7!}\int_a^b x^7 dx=\frac{1}{40320}(b^8-a^8)$;

By using (1.4) $\Longrightarrow\ =\frac{b-a}{7!\,.768}(97a^7+91a^6b+105a^5b^2+91a^4b^3+91a^3b^4+105a^2b^5+91ab^6+97ab^7)(a+b)-4(b-a)^6ab)$,

Hence,

$$E_{12}[f]=\frac{1}{40320}(b^8-a^8)=\frac{b-a}{7!\,.768}(97a^7+91a^6b+105a^5b^2+91a^4b^3+91a^3b^4+105a^2b^5+91ab^6+97ab^7)(a+b)-4(b-a)^6ab),\quad =-\frac{(b-a)^9}{3870720(a+b)}.$$

Therefore, the error term is

$$E_4[f]=-\frac{(b-a)^9}{3870720(a+b)}f^{(8)}(\xi).$$

V. CHMD-CNC FORMULAS

In this section, Contra-harmonic mean derivative-based closed Newton cotes quadrature formula is derived by using the Contra-harmonic mean value of the terminal points [a, b] at the derivative of Trapezoidal formula, Simpson's 1/3rd formula, Simpson's 3/8th formula and Boole's formula for the valuation of a definite integral.

Theorem 1.9

Closed Trapezoidal formula using Contra-harmonic mean derivative is

$\int_a^b f(x)dx \approx \frac{b-a}{2}[f(a)+f(b)] - \frac{(b-a)^3}{12} f^{(2)}\left(\frac{a^2+b^2}{a+b}\right),$

The precision of this method is 2.

Proof:

For $f(x) = x^2$ The exact value of $\int_a^b x^2 dx = \frac{1}{3}(b^3-a^3)$;(1.5)$\Longrightarrow -\left(\frac{b-a}{2}\right)(a^2-b^2) - \frac{2(b-a)^3}{12} = \frac{1}{3}(b^3-a^3)$.

It indicates that the solution is exact. Thus, the precision of the closed Trapezoidal formula with Contra-harmonic mean derivative is 2 whereas the precision of the existing Trapezoidal formula is 1.

Theorem 1.10

Closed Simpson's 1/3rd formula with Contra-harmonic mean derivative is

$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{(a+b)}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{a^2+b^2}{a+b}\right),$

The precision of this method is 4.

Proof:

For $f(x) = x^4$ The exact value of $\int_a^b x^4 dx = \frac{1}{5}(b^5-a^5)$; (1.6)$\Longrightarrow \left(\frac{b-a}{6}\right)\left[a^4 + 4\left(\frac{a+b}{2}\right)^4 + b^4\right] - \frac{24(b-a)^5}{2880} = \frac{1}{5}(b^5-a^5)$.

It indicates that the solution is exact. Thus, the precision of the closed Simpson's 1/3rd formula with Contra-harmonic mean derivative is 4 whereas the precision of the existing Simpson's 1/3rd formula is 3.

Theorem 1.11

Closed Simpson's 3/8th formula with Contra-harmonic mean derivative is

$\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^5}{6480} f^{(4)}\left(\frac{a^2+b^2}{a+b}\right),$

The precision of this method is 4.

Proof:

For $f(x) = x^4$ The exact value of $\int_a^b x^4 dx = \frac{1}{5}(b^5-a^5)$; (1.7)$\Longrightarrow \left(\frac{b-a}{8}\right)\left[a^4 + 3\left(\frac{2a+b}{3}\right)^4 + 3\left(\frac{a+2b}{3}\right)^4 + b^4\right] - \frac{24(b-a)^5}{6480} = \frac{1}{5}(b^5-a^5)$.

It indicates that the solution is exact. Thus, the precision of the closed Simpson's 3/8th formula with Contra-harmonic mean derivative is 4 whereas the precision of the existing Simpson's 3/8th formula is 3.

Theorem 1.12

Closed Boole's formula with Contra-harmonic mean derivative is

$\int_a^b f(x)dx \approx \frac{b-a}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right] - \frac{(b-a)^7}{1935360} f^{(6)}\left(\frac{a^2+b^2}{a+b}\right),$

The precision of this method is 6.

Proof:

For $f(x) = x^6$ The exact value of $\int_a^b x^6 dx = \frac{1}{7}(b^7-a^7)$; (1.8)$\Longrightarrow \left(\frac{b-a}{90}\right)\left[7a^6 + 32\left(\frac{3a+b}{4}\right)^6 + 12f\left(\frac{a+b}{2}\right)^6 + 32\left(\frac{a+3b}{4}\right)^6 + 7b^6\right] + \frac{720(b-a)^7}{1935360} = \frac{1}{7}(b^7-a^7)$.

It indicates that the solution is exact. Thus, the precision of the closed Boole's formula with Contra-harmonic mean derivative is 6 whereas the precision of the existing Boole's formula is 5.

VI. THE ERROR TERMS OF CHMD-CNC FORMULAS

In this section, the error term for the CHMD-CNC formula is derived by using the remainder between the quadrature formula for the monomial $\frac{x^{p+1}}{(p+1)!}$ and the exact result $\frac{1}{(p+1)!}\int_a^b x^{p+1}\,dx$ where p is the precision of the quadrature formula.

Theorem 1.13

Contra-harmonic mean derivative-based closed Trapezoidal formula with the error term is

$\int_a^b f(x)dx \approx \frac{b-a}{2}\big(f(a)+f(b)\big) - \frac{(b-a)^3}{12} f^{(2)}\left(\frac{a^2+b^2}{a+b}\right) + \frac{(b-a)^5}{24(a+b)} f^{(4)}(\xi).$ (1.13)

where $\xi \in$ (a, b). This is the fourth order accurate with the error term

$E_{13}[f] = -\frac{(b-a)^5}{24(a+b)} f^{(4)}(\xi).$

Proof:

The exact value of $\frac{1}{3!}\int_a^b x^3 dx = \frac{1}{24}(b^4-a^4)$;

By using (3.5) $\Longrightarrow \frac{b-a}{3!\,.2}\left(b^3 + a^3 - (b-a)^2\left(\frac{a^2+b^2}{a+b}\right)\right)$

Hence,

$E_{13}[f] = -\frac{1}{24}(b^4-a^4) - \frac{b-a}{3!\,.2}\left(b^3 + a^3 - (b-a)^2\left(\frac{a^2+b^2}{a+b}\right)\right) = -\frac{(b-a)^5}{24(a+b)}.$

Therefore, the error term is

$E_{13}[f] = -\frac{(b-a)^5}{24(a+b)} f^{(4)}(\xi).$

Theorem 1.14

Contra-harmonic mean derivative-based closed Simpson's 1/3rd formula with the error term is

$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{a^2+b^2}{a+b}\right) - \frac{(b-a)^7}{5760(a+b)} f^{(6)}(\xi).$ (1.14)

where $\xi \in$ (a, b). This is the fourth order accurate with the error term

$E_{14}[f] = -\frac{(b-a)^7}{5760(a+b)} f^{(6)}(\xi).$

Proof:

The exact value of $\frac{1}{5!}\int_a^b x^5 dx = \frac{1}{720}(b^6-a^6)$;

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By using (1.6) $\Rightarrow \frac{b-a}{5!\,.48}\left(8a^5+(a+b)^5+8b^5-2(b-a)^4\left(\frac{a^2+b^2}{a+b}\right)\right)$

Hence,

$$E_{14}[f]=-\frac{1}{720}(b^6-a^6)-\frac{b-a}{5!\,.48}\left(8a^5+(a+b)^5+8b^5-2(b-a)^4\left(\frac{a^2+b^2}{a+b}\right)\right)=-\frac{(b-a)^7}{5760(a+b)}.$$

Hence, the error term is

$$E_{14}[f]=-\frac{(b-a)^7}{5760(a+b)}f^{(6)}(\xi).$$

Theorem 1.15

Contra-harmonic mean derivative-based closed Simpson's 3/8th formula with the error term is

$$\int_a^b f(x)dx\approx\frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{(b-a)^5}{6480}f^{(4)}\left(\frac{a^2+b^2}{a+b}\right)-\frac{(b-a)^7}{12960(a+b)}f^{(6)}(\xi).\quad(1.15)$$

where $\xi\,\epsilon$ (a, b). This is the fourth order accurate with the error term

$$E_{15}[f]=-\frac{(b-a)^7}{12960(a+b)}f^{(6)}(\xi).$$

Proof:

The exact value of $\frac{1}{5!}\int_a^b x^5dx=\frac{1}{720}(b^6-a^6)$;

By using (1.7) $\Rightarrow\frac{b-a}{5!\,.648}\left(81a^5+(2a+b)^5+(a+2b)^5+81b^5-12(b-a)^4\left(\frac{a^2+b^2}{a+b}\right)\right)$

Hence,

$$E_{15}[f]=-\frac{1}{720}(b^6-a^6)-\frac{b-a}{5!\,.648}\begin{pmatrix}81a^5+(2a+b)^5+(a+2b)^5+81b^5\\-12(b-a)^4\left(\frac{a^2+b^2}{a+b}\right)\end{pmatrix}=-\frac{(b-a)^7}{12960(a+b)}.$$

Hence, the error term is

$$E_{15}[f]=-\frac{(b-a)^7}{12960(a+b)}f^{(6)}(\xi).$$

Theorem 3.16

Contra-harmonic mean derivative-based closed Boole's formula with the error term is

$$\int_a^b f(x)dx\approx\frac{b-a}{90}\left[7f(a)+32f\left(\frac{3a+b}{4}\right)+12f\left(\frac{a+b}{2}\right)+32f\left(\frac{a+3b}{4}\right)+7f(b)\right]-\frac{(b-a)^7}{1935360}f^{(6)}\left(\frac{a^2+b^2}{a+b}\right)+\frac{(b-a)^9}{3870720(a+b)}f^{(8)}(\xi).\quad(3.16)$$

where $\xi\,\epsilon$ (a, b). This is the fourth order accurate with the error term

$$E_{16}[f]=\frac{(b-a)^9}{3870720(a+b)}f^{(8)}(\xi).$$

Proof:

The exact value of $\frac{1}{7!}\int_a^b x^7dx=\frac{1}{40320}(b^6-a^6)$;

By using (1.8) $\Rightarrow$

$$\frac{b-a}{7!\,.748}\begin{pmatrix}97a^7+91a^6b+105a^5b^2+91a^4b^3+91a^3b^4\\+105a^2b^5+91ab^6+97ab^7-2(b-a)^4\left(\frac{a^2+b^2}{a+b}\right)\end{pmatrix}$$

Hence,

$$E_{16}[f]=-\frac{1}{720}(b^6-a^6)-\frac{b-a}{7!\,.648}\begin{pmatrix}97a^7+91a^6b+105a^5b^2+91a^4b^3+91a^3b^4\\+105a^2b^5+91ab^6+97ab^7-2(b-a)^4\left(\frac{a^2+b^2}{a+b}\right)\end{pmatrix}=\frac{(b-a)^9}{3870720(a+b)}.$$

Therefore, the error term is

$$E_{16}[f]=\frac{(b-a)^9}{3870720(a+b)}f^{(8)}(\xi).$$

VII. SUMMARY

The summary of precision, the orders and the error terms of Harmonic mean and Contra-harmonic mean derivative-based CNC formulas are presented in Table 1.1.

Table 1.1: The Error terms of HMD and CHMD based CNC formulas

Formulas	HMD-CNC			CHMD-CNC		
	Precision	Order	Error terms	Precision	Order	Error terms
Trapezoidal	2	5	$-\frac{(b-a)^5}{24(a+b)}f^{(4)}(\xi)$	2	5	$\frac{(b-a)^5}{24(a+b)}f^{(4)}(\xi)$
Simpson's 1/3rd	4	7	$-\frac{(b-a)^7}{5760(a+b)}f^{(6)}(\xi)$	4	7	$\frac{(b-a)^7}{5760(a+b)}f^{(6)}(\xi)$
Simpson's 3/8th	4	7	$-\frac{(b-a)^7}{12960(a+b)}f^{(6)}(\xi)$	4	7	$\frac{(b-a)^7}{12960(a+b)}f^{(6)}(\xi)$
Boole's	6	9	$-\frac{(b-a)^9}{3870720(a+b)}f^{(8)}(\xi)$	6	9	$\frac{(b-a)^9}{3870720(a+b)}f^{(8)}(\xi)$

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Table 3.1 indicate that the proposed HMD and CHMD based CNC formulas increase a single order of precision than the existing CNC formulas.

1.8 Numerical examples

To compare the efficiency of the existing CNC formulas and the HMD and CHMD based CNC formulas.

Example 1.1

Solve $\int_1^2 e^x dx$ and compare the solutions with the CNC and HMD-CNC formulas.

Solution:

The exact value of $\int_1^2 e^x dx$ = 4.67077427

Table 1.2: Comparison of CNC and HMD-CNC formulas - Example 1.1

Formulas	CNC		HMD-CNC	
	Approximate Value	Error	Approximate Value	Error
Trapezoidal	5.053668964	0.382894694	4.737529973	0.066755703
Simpson's 1/3rd	4.672349035	0.001574765	4.671031784	0.000257519
Simpson's 3/8th	4.671476470	0.000702200	4.670891027	0.000116757
Boole's	4.670776607	0.000002337	4.670774647	0.000000371

Example 1.2

Solve $\int_1^2 \frac{dx}{1+x}$ and compare the solutions with the CNC and HMD-CNC formulas.

Solution:

The exact value of $\int_1^2 \frac{dx}{1+x}$ = 0.405465108

Table 1.3: Comparison of CNC and HMD-CNC formulas - Example 1.2

Formulas	CNC		HMD-CNC	
	Approximate Value	Error	Approximate Value	Error
Trapezoidal	0.416666667	0.011201559	0.403547132	0.001917976
Simpson's 1/3rd	0.405555556	0.000090448	0.405435069	0.000030039
Simpson's 3/8th	0.405505952	0.000040844	0.405452402	0.000012706
Boole's	0.405465768	0.000000660	0.405464780	0.000000328

Example 1.3

Solve $\int_0^1 3^x$ and compare the answers with the CNC and CHMD-CNC formulas.

Solution:

The exact value of $\int_0^1 3^x$ =1.820478453.

Table 1.4: Comparison of CNC and CHMD-CNC formulas – Example 1.3

Formulas	CNC		CHMD-CNC	
	Approximate Value	Error	Approximate Value	Error
Trapezoidal	2.000000000	0.179521547	1.698262760	0.122215693
Simpson's 1/3rd	1.821367205	0.000888752	1.819849782	0.000628671

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Simpson's 3/8th	1.820875023	0.000396570	1.820200613	0.000277840
Boole's	1.820480044	0.000001591	1.820477319	0.000001134

Example 1.4

Solve $\int_1^2 e^x dx$ and compare the answers with the CNC and CHMD-CNC formulas.

Solution:

The exact value of $\int_1^2 e^x dx$ =4.67077427.

Table 1.5: Comparison of CNC and CHMD-CNC formulas - Example 1.4

Formulas	CNC		CHMD-CNC	
	Approximate Value	Error	Approximate Value	Error
Trapezoidal	5.053668964	0.382894694	4.612461460	0.058312810
Simpson's 1/3rd	4.672349035	0.001574765	4.670510670	0.000263600
Simpson's 3/8th	4.671476470	0.000702200	4.670659419	0.000114851
Boole's	4.670776607	0.000002337	4.670773871	0.000000399

VIII. CONCLUSION

In this chapter, Harmonic mean and Contra-harmonic mean derivative-based CNC formulas are derived for the evaluation of the definite integral with accurate results. The new error terms are also derived by using the difference between the exact result and the approximate solution. Numerical examples are also demonstrated for the accuracy of the proposed formulas.

REFERENCES

[1] C.T.H. Bakera and G.F. Miller, *Treatment of Integral Equations by Numerical Methods*, Academic Press Inc., London, 1982.

[2] T.K. Boehme, W. Preuss and W. van der Wall, *On a simple numerical method for computing Stieltjes integrals in reliability theory*, Probab. Engrg. Inform. Sci. 5, pp.113–128,1991.

[3] W.R.Boland and C.S.Duris, *Product type quadrature formulas*. BIT 11, pp.139-158, 1971.

[4] W.R.Boland, *Coefficients for product-type quadrature formula*, Math. Computation 26, 1972.

[5] C.Brezinski and L.Wuytack, *Numerical analysis in the twentieth century*, Elsevier science, pp.1 - 40, 2001.

[6] R. L. Burden and J. D. Faires, *Numerical Analysis*, Brooks/Cole, Boston, Mass, USA, 9th edition, 2011.

[7] Clarence O.E.Burg, *Derivative-based closed Newton-cotes numerical quadrature*, Applied Mathematics and Computations, Vol.*218*,pp.7052-7065, 2012.

[8] Clarence O.E.Burg and Ezechiel Degny, *Derivative-Based midpoint quadrature rule*, Applied mathematics, Vol.4, pp. 228- 234, 2013.

[9] P. Cerone and S. S. Dragomir, *Trapezoidal-type Rules from an Inequalities Point of View*, Handbook of Analytic-Computational Methods in Applied Mathematics, New York, pp.65–134, 2000.

[10] J. C. Chiou and S. D.Wu, *Open Newton-Cotes differential methods as multilayer symplectic integrators*, Journal of Chemical Physics, Vol. 107, no. 17, pp. 6894–6898, 1997.

[11] R. Cleroux And D. J. Mcconalogue, *A numerical algorithm for recursively-defined convolution integrals involving distribution functions*, Management Sci. 22, pp.1138-1146, 1976.

